Multiplication of 0-1 matrices via clustering*

Jesper Jansson[1], Mirosław Kowaluk[2], Andrzej Lingas[3], and Mia Persson[4]

[1] Graduate School of Informatics, Kyoto University, Kyoto, Japan.
jj@i.kyoto-u.ac.jp
[2] Institute of Informatics, University of Warsaw, Warsaw, Poland.
kowaluk@mimuw.edu.pl
[3] Department of Computer Science, Lund University, Lund, Sweden.
Andrzej.Lingas@cs.lth.se
[4] Department of Computer Science and Media Technology, Malmö University, Malmö, Sweden. Mia.Persson@mau.se

Abstract. We study applications of clustering (in particular, the k-center clustering problem) in the design of efficient and practical algorithms for computing an approximate and the exact arithmetic matrix product of two 0-1 rectangular matrices with clustered rows or columns, respectively. Our results in part can be regarded as an extension of the clustering-based approach to Boolean square matrix multiplication due to Arslan and Chidri (CSC 2011). First, we provide a simple and efficient deterministic algorithm for approximate matrix product of 0-1 matrices, where the additive error is proportional to the minimum maximum radius in an ℓ-center clustering of the rows of the first matrix or an k-center clustering of the columns of the second matrix. Next, we use the approximation algorithm as a preprocessing after which a query asking for the exact value of an arbitrary entry in the product matrix can be answered in time proportional to the additive error. As a consequence, we obtain a simple deterministic algorithm for the exact matrix product of 0-1 matrices. We also present an improved simple deterministic algorithm for the exact product and in addition, faster analogous randomized algorithms for an approximate and the exact matrix products of 0-1 matrices based on randomized ℓ and k-center clustering.

Keywords: arithmetic matrix multiplication, clustering, Hamming space, minimum spanning tree

1 Introduction

The arithmetic matrix product of two 0-1 matrices is closely related to the Boolean one of the corresponding Boolean matrices. Both are basic tools in science and engineering. For square $n \times n$ matrices, both can be computed in

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$O(n^{2.372})$ time by using fast matrix multiplication algorithms based on algebra [2, 32]. Unfortunately, the latter algorithms suffer from huge overheads and no truly subcubic practical algorithms for any of these two problems is known. Therefore, many researchers studied the complexity of these products for special input matrices, e.g., sparse or structured matrices [5, 4, 7, 16, 21, 30, 33], providing faster and often more practical algorithms.

The method of multiplying matrices with clustered rows or columns, proposed for Boolean matrix product in [7] and subsequently generalized in [16, 21] and used in [3], relies on the construction of an approximate minimum spanning tree of the rows of the first input matrix or the columns of the second input matrix in a Hamming space. Then, each column or each row of the product matrix is computed with the help of a traversal of the tree in time proportional to the total Hamming cost of the tree up to a logarithmic factor. Simply, the next entry in a column or a row in the product matrix can be obtained from the previous one in time roughly proportional to the Hamming distance between the consecutive (in the tree traversal) corresponding rows or columns of the first or the second input matrix, respectively. Thus, in case the entire tree cost is substantially subquadratic in n, the total running time of this method becomes substantially subcubic provided that a good approximation of a minimum spanning tree of the rows of the first input matrix or the columns of the second one can be constructed in substantially subcubic time. As for simplicity and practicality, a weak point of this method is that in order to construct such an approximation relatively quickly, it employs a rather involved randomized algorithm.

Arslan and Chidri presented a simple algorithm for the Boolean matrix product of two input Boolean matrices with rows (in the first matrix) and columns (in the other matrix) within a small diameter d (in terms of the Hamming distance) in [6]. The basic idea is to compute the inner Boolean product of a representative of the rows of the first matrix and a representative of columns of the second matrix and then relatively cheaply recover the entries of the Boolean product matrix from the inner product. Their algorithm runs in $O(dn^2)$ time for $n \times n$ matrices. They also presented a generalization of their algorithm to include grouping the rows of the first matrix as well as the columns of the second matrix into k clusters of maximum radius within 2 of the minimum s. They incorporate Gonzalez's 2-approximation algorithm for the k-center problem [22] to achieve such a k-clustering. Their general algorithm runs in $O((k+s)n^2)$ time[5].

In case of the arithmetic matrix product of 0-1 matrices, in some cases, a faster approximate arithmetic matrix multiplication can be more useful [11, 30]. Among other things, it can enable to identify largest entries in the product matrix and it can be also used to provide a fast estimation of the number of: the so called witnesses for the Boolean product of two Boolean matrices [20],

[5] They also claim that one can speed up their general algorithm by using the fast 2-approximation algorithm for the k-center problem due to Feder and Greene [14] instead of Gonzalez's one. Unfortunately, the former algorithm seems to have rather an exponential hidden dependence on the dimension (in this case n) in view of Lemmata 4.2 and 4.3 in [14] than a linear one as it is assumed in [6].

triangles in a graph, or more generally, subgraphs isomorphic to a small pattern graph [17] etc. There is a number of results on approximate arithmetic matrix multiplication, where the quality of approximation is expressed in terms of the Frobenius matrix norm $|| \; ||_F$ (i.e., the square root of the sum of the squares of the entries of the matrix) [11, 30].

Cohen and Lewis [11] and Drineas *et al.* [12] used random sampling to approximate arithmetic matrix product. Their articles provide an approximation D of the matrix product AB of two $n \times n$ matrices A and B such that $||AB - D||_F = O(||AB||_F/\sqrt{c})$, for a parameter $c > 1$ (see also [30]). The approximation algorithm in [12] runs in $O(n^2c)$ time. Drineas et al. [12] also derived bounds on the entrywise differences between the exact matrix product and its approximation. Unfortunately, the best of these bounds is $\Omega(M^2n/\sqrt{c})$, where M is the maximum value of an entry in A and B. By using a sketch technique, Sarlós [31] obtained the same Frobenius norm guarantees, also in $O(n^2c)$ time. However, he derived stronger individual upper bounds on the additive error of each entry D_{ij} of the approximation matrix D. They are of the form $O(||A_{i*}||_2||B_{*j}||_2/\sqrt{c})$, where A_{i*} and B_{*j} stands for the i-th row of A and the j-th column of B, respectively, that hold with high probability. More recently, Pagh [30] presented a randomized approximation $\widetilde{O}(n(n+c))$-time algorithm for the arithmetic product of $n \times n$ matrices A and B such that each entry of the approximate matrix product differs at most by $||AB||_F/\sqrt{c}$ from the correct one. His algorithm first compresses the matrix product to a product of two polynomials and then uses the fast Fourier transform to multiply the polynomials. Subsequently, Kutzkov [29] developed analogous deterministic algorithms employing different techniques. For approximation results related to sparse arithmetic matrix products, see [25, 30].

1.1 Our contributions

In this article, we exploit the possibility of applying the classic simple 2-approximation algorithm for the k center clustering problem [22] in order to derive efficient and practical deterministic algorithms for computing an approximate and the exact arithmetic matrix product of two 0-1 rectangular matrices A and B with clustered rows or columns, respectively. In addition, we consider also the possibility of applying the recent faster randomized algorithm for k center clustering from [28] instead of that from [22].

Our results are based on the idea of using cluster representatives/centers to compute faster a smaller matrix product and then relatively cheaply updating the small product to the target one. This natural idea has been used already in [6][6]. There are however several differences between our approach and that in [6]. We consider the more general problem of computing the arithmetic matrix product of 0-1 rectangular matrices while [6] is concerned with the Boolean matrix product of square Boolean matrices. We provide also cluster-based approximation and

[6] We came across this idea independently without knowing [6] before writing the preliminary conference version of this article.

preprocessing for a query on the value of single entry of the aforementioned arithmetic product. In [6], the rows of the first matrix are grouped in the same number of clusters as the columns of the second matrix, while in our case the numbers are in general different. Also, in [6] representatives/centers of both row clusters and column clusters are used to compute the smaller product while we use in the basic deterministic version either the row ones or the column ones.

The k-center clustering problem in a Hamming space $\{0,1\}^d$ is for a set P of n points in $\{0,1\}^d$ to find a set T of k points in $\{0,1\}^d$ that minimize $\max_{v\in P}\min_{u\in T}\mathrm{ham}(v,u)$, where $\mathrm{ham}(v,u)$ stands for the Hamming distance between v and u (i.e., the number of coordinates where v and u differ). Each center in T induces a cluster consisting of all points in P for which it is the nearest center.

Let $\lambda(A,\ell,row)$ and $\lambda(B,k,col)$ denote the minimum maximum Hamming distance between a row of A or a column of B and the closest center in an ℓ-center clustering of the rows of A or in a k-center clustering of the columns of B, respectively. In particular, when A and B are square matrices of size $n\times n$, we obtain the following results.

1. A simple deterministic algorithm that approximates each entry of the arithmetic matrix product of A and B within an additive error of at most $2\lambda(A,\ell,row)$ in $O(n^2\ell)$ time or at most $2\lambda(B,k,col)$ in $O(n^2k)$ time. Similarly, a simple randomized algorithm that for any $\epsilon\in(0,\frac{1}{2})$ approximates each entry of the product within an additive error of with high probability (w.h.p.) at most $(2+\epsilon)(\lambda(A,\ell,row)+\lambda(B,k,col))$ in time $O(n\log n(n+\ell+k)/\epsilon^2+\ell nk)$.
2. A simple deterministic preprocessing of the matrices A and B in $O(n^2\ell)$ time or $O(n^2k)$ time after which every query asking for the exact value of an arbitrary entry of the arithmetic matrix product of A and B can be answered in $O(\lambda(A,\ell,row))$ time or $O(\lambda(B,k,col))$ time, respectively. Similarly, a simple randomized preprocessing of the matrices in time $O(n\log n(n+\ell+k)+\ell nk)$ after which every query asking for the exact value of an arbitrary entry of the product of the matrices can be answered in $O(\lambda(A,\ell,row)+\lambda(B,k,col))$ time w.h.p.
3. A simple deterministic algorithm for the exact arithmetic matrix product of A and B running in time $O(n^2(\ell+k+\min\{\lambda(A,\ell,row),\lambda(B,k,col)\}))$. Alternatively, a simple randomized algorithm for the exact product running in time $O(n\log n(n+\ell+k)+\ell nk+n^2(\lambda(A,\ell,row)+\lambda(B,k,col))$ w.h.p.

1.2 Techniques

All our main deterministic results rely on the classical, simple 2-approximation algorithm for the k-center clustering problem (farthest-point clustering) due to Gonzalez [22], whose properties are summarized in Fact 1 below. Our results also rely on the idea of updating the inner product of two vectors a and b in $\{0,1\}^q$ over the Boolean or an arithmetic semi-ring to that of two vectors a' and b' in $\{0,1\}^q$, in time roughly proportional to $\mathrm{ham}(a,a')+\mathrm{ham}(b,b')$. The idea has been used in [6, 7, 16, 21]. As in some of the aforementioned articles, we combine it with a traversal of an approximate minimum spanning tree of the rows or columns of an input matrix in the Hamming space $\{0,1\}^q$, where q is the length of the rows or columns (see Lemma 8).

1.3 Article organization

The next section contains basic definitions. Section 3 presents our approximation algorithms for the arithmetic product of two 0-1 matrices and the preprocessing enabling efficient answers to queries asking for the value of an arbitrary entry of the arithmetic product matrix. Section 4 is devoted to our algorithms for the exact arithmetic matrix product of two 0-1 matrices. We conclude with a short discussion on potential extensions of our results.

2 Preliminaries

For a positive integer r, $[r]$ stands for the set of positive integers not exceeding r.

For two vectors $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ in $\mathbb{R}^d$, their *inner product* is equal to $\sum_{\ell=1}^{d} a_\ell b_\ell$. The transpose of a matrix D is denoted by $D^\top$. If the entries of D are in $\{0,1\}$ then D is a 0-1 matrix.

The *Hamming distance* between two points a, b (vectors) in $\{0,1\}^d$ is the number of coordinates in which the two points differ. Alternatively, it can be defined as the distance between a and b in the L_1 metric over $\{0,1\}^d$. It is denoted by $\text{ham}(a,b)$.

An event is said to hold *with high probability* (w.h.p. for short) in terms of a parameter N related to the input size if it holds with probability at least $1 - \frac{1}{N^\alpha}$, for any constant α not less than 1.

The *k-center clustering problem in a Hamming space* is as follows: Given a set P of n points in a Hamming space $\{0,1\}^d$, find a set T of k points in $\{0,1\}^d$ that minimize $\max_{v\in P}\min_{u\in T}\text{ham}(v,u)$. It is known to be NP-hard already for $k = 1$ [18] and also NP-hard to approximate within a ratio of $2-\epsilon$ for any constant $\epsilon > 0$ when k is part of the input [14, 19]. Note that according to the problem definition, the points in T do not have to belong to the set P. The *minimum-diameter k-clustering problem in a Hamming space* is: Given a set P of n points in a Hamming space $\{0,1\}^d$, find a partition of P into k subsets $P_1, P_2, \ldots, P_k$ that minimize $\max_{i\in[k]}\max_{v,u\in P_i}\text{ham}(v,u)$. The k-center clustering problem can also be termed the minimum-radius k-clustering problem.

Fact 1 *[22] (Gonzalez's algorithm) There is a simple deterministic 2-approximation algorithm for the k-center clustering and minimum-diameter k-clustering problems in a Hamming space running in $O(ndk)$ time. Moreover, the approximate solution returned by Gonzalez's algorithm consists entirely of points from the input set P.*

We will use the last property given in Fact 1 (that is, the approximate solutions found by Gonzalez's algorithm consist of points from the input only) in the proof of Theorem 5.

A combination of a variant of randomized dimension reduction in Euclidean spaces in the spirit of Johnson and Lindenstrauss [27] given by Achlioptas in [1] and the observation that the Hamming distance between two 0-1 vectors is equal to their squared L_2 distance makes it possible to decrease the dimension to a logarithmic one in the number of input points, at the cost of worsening the approximation guarantee by some arbitrarily small ϵ. Taking into account the time needed to compute the images of the input points in the subspace of logarithmic dimension, this yields the following fact whose full proof can be found in [28].

Fact 2 *[28] For any fixed $\epsilon \in (0, 1/2)$, there is a randomized algorithm for the k-center clustering problem in a Hamming space running in $O(n \log n(d + k)/\epsilon^2)$ time that w.h.p. computes a $(2+\epsilon)$-approximation of an optimal k-center clustering.*

3 An approximate arithmetic matrix product of 0-1 matrices

Our approximation algorithm for the arithmetic matrix product of two 0-1 matrices is specified by the following procedure.

procedure *MMCLUS-Approx*(A, B, ℓ)

Input: Two 0-1 matrices A and B of sizes $p \times q$ and $q \times r$, respectively, where $p \geq r$, and a positive integer ℓ not exceeding p.

Output: A $p \times r$ matrix D, where for $1 \leq i \leq p$ and $1 \leq j \leq r$, D_{ij} is an approximation of the inner product C_{ij} of the i-th row A_{i*} of A and the j-th column B_{*j} of B.

1. Determine an approximate solution to the ℓ-center clustering of the rows of the matrix A in $\{0, 1\}^q$. For each row A_{i*} of A, set $cen_\ell(A_{i*})$ to its closest center, with ties broken arbitrarily.
2. Form the $\ell \times q$ matrix A', where the i'-th row is the i'-th center in the approximate ℓ-center clustering of the rows of A.
3. Compute the arithmetic $\ell \times r$ matrix product C' of A' and B.
4. For $1 \leq i \leq p$ and $1 \leq j \leq r$, set D_{ij} to $C'_{i'j}$, where the i'-th row $A'_{i'*}$ of A' is $cen_\ell(A_{i*})$.

See Fig. 1 for an illustration of how C' is computed from A' and B in step 3. After having computed C', the algorithm then sets each entry D_{ij} of the approximate matrix D to the entry of C' holding the inner product of the center closest to the ith row and the jth column of B in step 4.

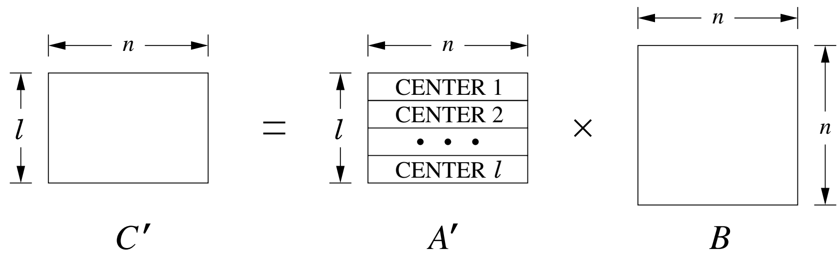

Fig. 1. Computing the matrix C' in step 3 of *MMCLUS-Approx*(A, B, ℓ).

For a 0-1 $p \times q$ matrix A, let $\lambda(A, \ell, row)$ be the minimum, over all ℓ-center clusterings of the rows of A in the Hamming space $\{0, 1\}^q$, of the maximum Hamming distance between a row of A and the closest center. Similarly, for a 0-1 $q \times r$ matrix B, let $\lambda(B, k, col)$ be the minimum, over all k-center clusterings of the columns of B in the Hamming space $\{0, 1\}^q$, of the maximum Hamming distance between a column of B and the closest center.

Lemma 1. *Suppose a 2-approximation algorithm for the ℓ-center clustering is used in MMCLUS-Approx(A, B, ℓ) and let C stand for the arithmetic product of A and B. Then, for $1 \le i \le p$ and $1 \le j \le r$, $|C_{ij} - D_{ij}| \le 2\lambda(A, \ell, row)$.*

Proof. Recall that $p \ge r$ is assumed in the input to *MMCLUS-Approx*(A, B, ℓ). For $1 \le i \le p$ and $1 \le j \le r$, D_{ij} is the inner product of $cen_\ell(A_{i*})$, where $\text{ham}(A_{i*}, cen_\ell(A_{i*})) \le 2\lambda(A, \ell, row)$, with B_{*j}. Hence, C_{ij}, which is the inner product of A_{i*} with B_{*j}, can differ at most by $2\lambda(A, \ell, row)$ from D_{ij}.

By $T(s, q, t)$, we shall denote the worst-case time taken by the multiplication of two 0-1 matrices of sizes $s \times q$ and $q \times t$, respectively.

Lemma 2. *MMCLUS-Approx(A, B, ℓ) can be implemented in $O(pq\ell + pr + T(\ell, q, r))$ time.*

Proof. Recall that $p \ge r$. Step 1, which includes the assignment of the closest center to each row of A, can be done in $O(pq\ell)$ time by using Fact 1. Step 2 takes $O(\ell q)$ time, which is $O(T(\ell, q, r))$ time. Finally, Step 3 takes $T(\ell, q, r)$ time while Step 4 can be done in $O(pr)$ time. Thus, the overall time is $O(pq\ell + pr + T(\ell, q, r))$.

We can use the straightforward $O(sqt)$-time algorithm for the multiplication of two matrices of sizes $s \times q$ and $q \times t$, respectively. Since $T(\ell, q, r) = O(\ell qr) = O(\ell qp)$ if $p \ge r$, Lemmata 1 and 2 yield the first part (1) of our first main result (Theorem 1 below), for $p \ge r$. Its second part (2) for $p \le r$ follows from the first part by $(AB)^\top = B^\top A^\top$. Note that then the number of rows in $B^\top$, which is r, is not less than the number of columns in $A^\top$, which is p. Simply, we run *MMCLUS-Approx*$(B^\top, A^\top, k)$ to compute an approximation of the transpose of the arithmetic matrix product of A and B. Note also that a k-clustering of columns of B is equivalent to a k-clustering of the rows of $B^\top$ and hence $\lambda(B^\top, k, row) = \lambda(B, k, col)$.

Theorem 1. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. There is a simple deterministic algorithm which provides an approximation of all entries of the arithmetic matrix product of A and B within an additive error of at most:*

1. $2\lambda(A, \ell, row)$ *in time* $O(pq\ell + pr)$ *if* $p \geq r$,
2. $2\lambda(B, k, col)$ *in time* $O(rqk + pr)$ *if* $p \leq r$.

The heavy three-linear terms $pq\ell$ and rqk in the upper time bounds in Theorem 1 originate from the upper time bound on the center clustering problem in Fact 1 and the time taken by matrix multiplication. We can get rid of these terms by using a faster randomized algorithm for approximate center clustering according to Fact 2, both on the rows of the first matrix A and the columns of the second matrix B, in a way similar to [6]. Then, only the small matrices induced by the centers of the rows of A and the columns of B, respectively, need to be multiplied. In this way, the terms $pq\ell$ and rqk are replaced by the presumably smaller term ℓqk in the upper time bound at the cost of increasing the additive error.

procedure *MMCLUS-R-Approx*$(A, B, \ell, k, \epsilon)$

Input: Two 0-1 matrices A and B of sizes $p \times q$ and $q \times r$, respectively, positive integers ℓ, k not exceeding p, r, respectively, and a real number $\epsilon \in (0, 1/2)$.

Output: A $p \times r$ matrix D, where for $1 \leq i \leq p$ and $1 \leq j \leq r$, D_{ij} is an approximation of the inner product C_{ij} of the i-th row A_{i*} of A and the j-th column B_{*j} of B.

1. Determine an approximate ℓ-center clustering of the rows of the matrix A in $\{0,1\}^q$ by using Fact 2. For each row A_{i*} of A, set $cen_\ell(A_{i*})$ to its closest center, with ties broken arbitrarily.
2. Determine an approximate k-center clustering of the columns of the matrix B in $\{0,1\}^q$ by using Fact 2. For each column B_{*j} of B, set $cen_k(B_{*j})$ to its closest center, with ties broken arbitrarily.
3. Form the $\ell \times q$ matrix A', where the i'-th row is the i'-th center in the approximate ℓ-center clustering of the rows of A.
4. Form the $q \times k$ matrix B', where the j'-th row is the j'-th center in the approximate k-center clustering of the columns of B.
5. Compute the arithmetic $\ell \times k$ matrix product C'' of A' and B'.
6. For $1 \leq i \leq p$ and $1 \leq j \leq r$, set D'_{ij} to $C''_{i'j'}$, where the i'-th row $A'_{i'*}$ of A' is $cen_\ell(A_{i*})$ and the j'-th column $B'_{*j'}$ of B' is $cen_k(B_{*j})$.

Lemma 3. *Let C stand for the arithmetic product of A and B. For $1 \leq i \leq p$ and $1 \leq j \leq r$, $|C_{ij} - D'_{ij}| \leq (2+\epsilon)(\lambda(A, \ell, row) + \lambda(B, k, col))$ holds.*

Proof. For $1 \leq i \leq p$ and $1 \leq j \leq r$, D'_{ij} is the inner product of $cen_\ell(A_{i*})$, where $\text{ham}(A_{i*}, cen_\ell(A_{i*})) \leq (2+\epsilon)\lambda(A, \ell, row)$, with $cen_k(B_{*j})$, where ham$(B_{*j}, cen_k(B_{*j})) \leq (2+\epsilon)\lambda(B,k,col)$ by Fact 2. Hence, C_{ij}, which is the inner product of A_{i*} with B_{*j}, can differ at most by $(2+\epsilon)(\lambda(A,\ell,row) + \lambda(B,k,col))$ from D'_{ij}.

Lemma 4. *MMCLUS-R-Approx*(A,B,ℓ,k,ϵ) *can be implemented in time* $O(p\log p(q+\ell)/\epsilon^2 + r\log r(q+k)/\epsilon^2 + pr + T(\ell,q,k))$.

Proof. In Step 1, the ℓ centers of the rows of A can be found in $O(p\log p(q+\ell)/\epsilon^2)$ time by Fact 2. Also, the ℓ clusters induced by the centers can be formed in $O(p\log p(q+\ell)/\epsilon^2)$ time in order to approximate minimum-diameter ℓ-clustering of the rows of A; see the proof of Fact 2 in [28]. Importantly, each member of such a cluster is within a Hamming distance not exceeding $(2+\epsilon)\lambda(A,\ell,row)$ from its center. Symmetrically, Step 2 takes $O(r\log r(q+k)/\epsilon^2)$ time. Steps 3 and 4 can be done $O(\ell q)$ time and $O(kq)$ time, respectively, which is $O(T(\ell,q,k))$ time. Finally, Step 5 takes $T(\ell,q,k)$ time while Step 6 can be done in $O(pr)$ time. Thus, the overall time is $O(p\log p(q+\ell)/\epsilon^2 + r\log r(q+k)/\epsilon^2 + pr + T(\ell,q,k))$.

Lemmata 3 and 4 yield our second theorem.

Theorem 2. *Let A and B be two 0-1 matrices of sizes $p\times q$ and $q\times r$, respectively. There is a simple randomized algorithm which for any $\epsilon \in (0,1/2)$ provides an approximation of all entries of the arithmetic matrix product of A and B within an additive error of w.h.p. at most* $(2+\epsilon)(\lambda(A,\ell,row)+\lambda(B,k,col))$ *in time* $O(p\log p(q+\ell)/\epsilon^2 + r\log r(q+k)/\epsilon^2 + \ell qk + pr)$.

We can apply *MMCLUS-Approx*(A,B,ℓ) to obtain a preprocessing for answering queries about single entries of the arithmetic matrix product of A and B.

procedure *MMCLUS-Preproc*(A,B,ℓ)

Input: Two 0-1 matrices A and B of sizes $p\times q$ and $q\times r$, respectively, where $p \geq r$, and a positive integer ℓ not exceeding p.

Output: The $p\times r$ matrix D returned by *MMCLUS-Approx*(A,B,ℓ), and for $1\leq i\leq p$, the set of coordinate indices $ind(A,i)$ on which A_{i*} differs from the closest center.

1. Run *MMCLUS-Approx*(A,B,ℓ).
2. For $1\leq i\leq p$, determine the set $ind(A,i)$ of coordinate indices on which A_{i*} differs from $cen_\ell(A_{i*})$.

Lemma 5. *MMCLUS-Preproc*(A,B,ℓ) *can be implemented in* $O(pq\ell + pr + T(\ell,q,r))$ *time.*

Proof. Recall that $p\geq r$. Step 1 can be done in $O(pq\ell + pr + T(\ell,q,r))$ time by Lemma 2. Step 2 can easily be implemented in $O(pq)$ time.

Our procedure for answering a query about a single entry of the matrix product of A and B is as follows.

procedure $MMCLUS$-$Query(A, B, \ell, i, j)$

Input: The preprocessing done by $MMCLUS$-$Preproc(A, B, \ell)$ for 0-1 matrices A and B of sizes $p \times q$ and $q \times r$, respectively, where $p \geq r$, $\ell \in [p]$, and two query indices $i \in [p]$ and $j \in [r]$.

Output: The inner product C_{ij} of the i-th row A_{i*} of A and the j-th column B_{*j} of B.

1. Set C_{ij} to the entry D_{ij} of the matrix D computed by $MMCLUS$-$Approx(A, B, \ell)$ in $MMCLUS$-$Preproc(A, B, \ell, k)$.
2. For $m \in ind(A, i)$ do
   (a) If the m-th coordinate of the center assigned to A_{i*} is 0 and $B_{mj} = 1$ then $C_{ij} \leftarrow C_{ij} + 1$.
   (b) If the m-th coordinate of the center assigned to A_{i*} is 1 and B_{mj} is also 1 then $C_{ij} \leftarrow C_{ij} - 1$.

Lemma 6. *$MMCLUS$-$Query(A, B, \ell, i, j)$ is correct, i.e., the final value of C_{ij} is the inner product of the i-th row A_{i*} of A and the j-th column B_{*j} of B.*

Proof. C_{ij} is initially set to D_{ij}, which is the inner product of the center assigned to A_{i*} and B_{*j}. Then, C_{ij} is appropriately corrected by increasing or decreasing with 1 for each coordinate index $m \in ind(A, i)$ which contributes 1 to the inner product of A_{i*} and B_{*j} and 0 to the inner product of the center of A_{i*} and B_{*j} or *vice versa*.

Lemma 7. *$MMCLUS$-$Query(A, B, \ell, i, j)$ takes $O(\lambda(A, \ell, row))$ time.*

Proof. Recall that $2\lambda(A, \ell, row)$ is an upper bound on the maximum Hamming distance between a row of A and the closest center in the ℓ-center clustering computed by $MMCLUS$-$Approx(A, B, \ell)$ in $MMCLUS$-$Preproc(A, B, \ell)$. Recall also that $p \geq r$. Step 1 takes $O(1)$ time. Since the m-th coordinate in the centers can be accessed in the matrix A' computed by $MMCLUS$-$Approx(A, B, \ell)$, each of the two substeps in the block of the loop in Step 2 can be done in $O(1)$ time. Finally, since $|ind(A, j)| \leq 2\lambda(A, \ell, row)$, the block is iterated at most $2\lambda(A, \ell, row)$ times. Consequently, the whole Step 2 takes $O(\lambda(A, \ell, row))$ time.

By putting Lemmata 5, 6, and 7 together, and using the straightforward $O(sqt)$-time algorithm to multiply matrices of size $s \times q$ and $q \times t$, we obtain our next main result for $p \geq r$. The case $p \leq r$ reduces to the case $p \geq r$ by $(AB)^\top = B^\top A^\top$. Recall that then the number of rows in $B^\top$, which is r, is not less than the number of columns in $A^\top$, which is p. Also, we have $\lambda(B^\top, k, row) = \lambda(B, k, col)$. We simply run $MMCLUS$-$Preproc(B^\top, A^\top, k)$ and $MMCLUS$-$Query(B^\top, A^\top, k, j, i)$ instead.

Theorem 3. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given parameters $\ell \in [p]$ and $k \in [r]$, the matrices can be preprocessed by a simple deterministic algorithm in $O(pq\ell + pr)$ time if $p \geq r$ or $O(rqk + pr)$ time if $p \leq r$ such that a query asking for the exact value of a single entry C_{ij} of the arithmetic matrix product C of A and B can be answered in $O(\lambda(A, \ell, row))$ time if $p \geq r$ or $O(\lambda(B, k, col))$ time if $p \leq r$.*

Analogously, by using Theorem 2 with ϵ set, say, to $\frac{1}{4}$ instead of Theorem 1, we obtain the following randomized variant of Theorem 3.

Theorem 4. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given parameters $\ell \in [p]$ and $k \in [r]$, the matrices can be preprocessed by a simple randomized algorithm in time $O(p \log p(q+\ell) + r \log r(d+k) + \ell qk + pr)$ such that a query asking for the exact value of a single entry C_{ij} of the arithmetic matrix product C of A and B can be answered in $O(\lambda(A, \ell, row) + \lambda(B, k, col))$ time w.h.p.*

4 The exact arithmetic matrix product of 0-1 matrices

Theorem 3 yields the following corollary.

Corollary 1. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given parameters $\ell \in [p]$ and $k \in [r]$, the arithmetic matrix product of A and B can be computed by a simple deterministic algorithm in time $O(pq\ell + pr\lambda(A, \ell, row))$ if $p \geq r$ or time $O(rqk + pr\lambda(B, k, col))$ if $p \leq r$.*

Similarly, we obtain the following corollary from Theorem 4.

Corollary 2. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given parameters $\ell \in [p]$ and$k \in [r]$, the arithmetic matrix product of A and B can be computed by a simple randomized algorithm in time $O(p \log p(q + \ell) + r \log r(q + k) + \ell qk + pr(\lambda(A, \ell, row) + \lambda(B, k, col)))$ w.h.p.*

There is however a slightly better way of obtaining a simple deterministic algorithm for the arithmetic matrix product of two 0-1 matrices via ℓ-center clustering of the rows of the first matrix or k-center clustering of the columns of the second matrix than that of Corollary 1. The idea is to use the aforementioned technique of traversing an approximate minimum spanning tree of the rows of the first matrix or the columns of the second matrix in an appropriate Hamming space in order to compute a row or column of the product matrix [7, 16, 21]. The technique easily generalizes to 0-1 rectangular matrices. We shall use the following procedure and lemma in the spirit of [7, 16, 21].

procedure *MMCLUS-ST*(A, B, T)

Input: Two matrices A and B of sizes $p \times q$ and $q \times r$, respectively, and a spanning tree T of the rows of A in the Hamming space $\{0,1\}^q$.

Output: The arithmetic matrix product C of A and B.

1. Construct a traversal (i.e., a non-necessarily simple path visiting all vertices) U of T.
2. For any pair A_{m*}, A_{i*}, where the latter row follows the former in the traversal U, compute the set $diff(m,i)$ of indices $h \in [q]$ where $A_{ih} \neq A_{mh}$.
3. For $j = 1, \dots, r$ do:
   (a) Compute C_{sj} where A_{s*} is the row of A from which the traversal U of T starts.
   (b) While following U, do:
      i. Set $m,\ i$ to the indices of the previously traversed row of A and the currently traversed row of A, respectively.
      ii. Set C_{ij} to C_{mj}.
      iii. For each $h \in diff(m,i)$, if $A_{ih}B_{hj} = 1$ then set C_{ij} to $C_{ij} + 1$ and if $A_{mh}B_{hj} = 1$ then set C_{ij} to $C_{ij} - 1$.

Define the Hamming cost $\text{ham}(S)$ of a spanning tree S of a point set $P \subset \{0,1\}^d$ by $\text{ham}(S) = \sum_{(v,u)\in S} \text{ham}(v,u)$.

Lemma 8. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given a spanning tree T_A of the rows of A and a spanning tree T_B of the columns of B in the Hamming space $\{0,1\}^q$, the arithmetic matrix product of A and B can be computed in time $O(pq + qr + pr + \min\{r \times \text{ham}(T_A), p \times \text{ham}(T_B)\})$.*

Proof. First, we shall prove that *MMCLUS-ST*(A, B, T_A) computes the arithmetic matrix product of A and B in time $O(pq+qr+r\times\text{ham}(T_A))$. The correctness of the procedure follows from the correctness of the updates of C_{ij} in the block of the inner loop, i.e., in Step 3(b). Step 1 of *MMCLUS-ST*(A, B, T_A) can be done in $O(p)$ time while Step 2 requires $O(pq)$ time. The first step in the block under the outer loop, i.e., computing C_{sj} in Step 3(a), takes $O(q)$ time. The crucial observation is that the second step in this block, i.e., Step 3(b), requires $O(p+\text{ham}(T_A))$ time. Simply, the substeps (i), (ii) take $O(1)$ time while the substep (iii) requires $O(|diff(m,i)| + 1)$ time. Since the block is iterated r times, the whole outer loop, i.e., Step 3, requires $O(qr + pr + r\text{ham}(T_A))$ time. Thus, *MMCLUS-ST*(A, B, T_A) can be implemented in $O(pq+qr+rp+r\times\text{ham}(T_A))$ time.

Similarly, we can run *MMCLUS-ST*$(B^\top, A^\top, T_B)$ to obtain the transpose of the arithmetic matrix product of A and B. So, to obtain the lemma, we can alternate the steps of *MMCLUS-ST*(A, B, T_A) and *MMCLUS-ST*$(B^\top, A^\top, T_B)$, and stop whenever any of the calls is completed.

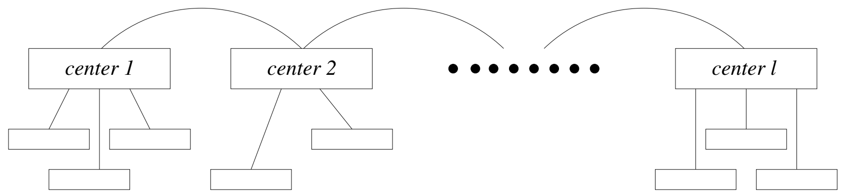

Fig. 2. The spanning tree T_A of the rows of A.

Theorem 5. *Let A and B be two 0-1 matrices of sizes $p \times q$ and $q \times r$, respectively. Given parameters $\ell \in [p]$ and $k \in [r]$, the arithmetic matrix product of A and B can be computed by a simple deterministic algorithm in time $O(pq\ell + rqk + \min\{pr\lambda(A,\ell,row) + rq\ell,\ pr\lambda(B,k,col) + pqk\})$.*

Proof. We determine an ℓ-center clustering of the rows of A in $\{0,1\}^q$ of maximum cluster radius not exceeding $2\lambda(A,\ell,row)$ in $O(pq\ell)$ time by employing Fact 1. Similarly, we construct a k-center clustering of the columns of B in $\{0,1\}^q$ of maximum cluster radius not exceeding $2\lambda(B,k,col)$ in $O(rqk)$ time. The centers in both aforementioned clusterings are some rows of A and some columns of B, respectively, by the last part of Fact 1. Hence, the ℓ-center clustering gives rise to a spanning tree T_A of the rows of A with all members of a cluster being pendants of their cluster center and the centers connected by a path of length $\ell-1$, see Fig. 2. The Hamming cost of T_A is at most $(p-\ell)2\lambda(A,\ell,row) + (\ell-1)q$. Similarly, we obtain a spanning tree T_B of the columns of B having the Hamming cost not exceeding $(r-k)2\lambda(B,k,col) + (k-1)q$. The theorem follows from Lemma 8 by straightforward calculations.

We note that in case $\ell = k$ and $\lambda(A,\ell,row) = \lambda(B,col)$, the upper time bounds in Corollary 1 and Theorem 5 coincide with that for the Boolean matrix product established in [6].

5 Potential extensions

The rows or columns in the input 0-1 matrices can be very long. Also, a large number of clusters might be needed in order to obtain a low upper bound on their radius. Among other things, for these reasons, we have picked Gonzalez's classical algorithm for the k-center clustering problem [22] as a basic tool in our deterministic approach to the arithmetic matrix product of two 0-1 matrices with clustered rows or columns. The running time of his algorithm is linear not only in the number of input points but also in their dimension, and in the parameter k. Importantly, it is very simple, deterministic, provides a solution within 2 of the optimum, and can be applied in Hamming spaces. For instance, there exist faster (in terms of n and k) 2-approximation algorithms for k-center clustering with hidden exponential dependence on the dimension in their running time, see

[14, 23]. Among several newer works on speeding approximation algorithms for k-center clustering (e.g., [13, 15, 26, 28]) only [28] explicitly includes Hamming spaces. The randomized approximation method for the k-center clustering problem in Hamming spaces from [28] is summarized in Fact 2. It provides approximation guarantees arbitrarily close to 2, is substantially faster than Gonzalez's algorithm when the dimension and k are superlogarithmic, and is relatively simple. For these reasons, our improved randomized algorithms for the approximate and exact 0-1 matrix multiplication are based on this method.

One could easily generalize our main results by replacing approximation algorithms for k-center clustering with approximation algorithms for the more general problem of k-center clustering with outliers [9]. In the latter problem, a given number z of input points could be discarded as outliers when trying to minimize the maximum cluster radius. Unfortunately, the algorithms for this more general problem tend to be more complicated and the focus seems to be on the approximation ratio achievable in polynomial time (e.g., 3 in [9] and 2 in [24]) and not on the time complexity.

There are many other variants of clustering than k-center clustering, and plenty of methods have been developed for them in the literature. In fact, in the design of efficient algorithms for the exact arithmetic matrix product of 0-1 matrices with clustered rows or columns, using the k-median clustering could seem more natural. The objective in the latter problem is to minimize the sum of distances between the input points and their nearest centers. Unfortunately, no simple deterministic $O(1)$-approximation algorithms for the latter problem that are efficient in case the dimension and k parameters are large seem be to available [8, 10].

Our approximate and exact algorithms for the matrix product of 0-1 matrices as well as the preprocessing of the matrices can be categorized as supervised since they assume that the user has some knowledge on the input matrices and can choose reasonable values of the parameters ℓ and k guaranteeing relatively low overall time complexity. Otherwise, one could try the ℓ-center and k-center clustering subroutines for a number of combinations of different values of ℓ and k in order to pick the combination yielding the lowest upper bound on the overall time complexity of the algorithm or preprocessing.

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